Faltings heights and subleading terms of adjoint L-functions

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Gross–Zagier 40 years later, Cambridge USA

“Arithmetic Lefschetz trace formula”?

K number field, $X \to \operatorname{Spec} K$ smooth proj., $\Delta \colon X \to X \times X$ diagonal

$T \colon X \dashrightarrow X$ finite étale correspondence $\qquad T \to X \times X$ graph

Count fixed points of T $\leftrightarrow (\Delta, T)$

Lefschetz trace/fixed point formula: if $H^*(X_{\overline{k}}) = \bigoplus_{\mathrm{Gal}(\overline{K}/K) \text{ irrep.} \rho} V_\rho$

$$(\Delta, T) = \sum_i (-1)^i \operatorname{tr}(T|_{H^i(X_{\overline{K}})}) = \sum_\rho \pm \operatorname{tr}(T|_{V_\rho})$$

$\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ sm. proj., $\hat{\Delta} \colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$, $\hat{T} \to \mathcal{X} \times \mathcal{X}$, $\widehat{\mathcal{L}}$ line bundle on $\mathcal{X}$

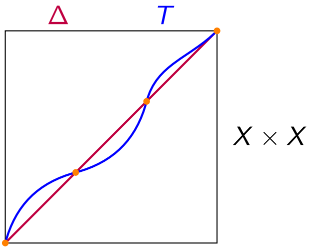
$$h(\Delta, T) \coloneqq (\hat{\Delta}, \hat{\mathcal{L}}, \hat{T}) = \operatorname{tr}(T|_{??}) \overset{??}{=} \sum_\rho \pm \operatorname{tr}(T|V_\rho) \cdot \boxed{\frac{L'(0,\pi_\rho,\mathrm{Ad})}{L(0,\pi_\rho,\mathrm{Ad})}}$$

For X a Shimura variety with $\rho \rightsquigarrow$ aut. rep. π_ρ

Hecke fixed points: count

$X = X_0(N)$ $\qquad X \xrightarrow{\Delta} X \times X$

Hecke operator $\qquad T \longrightarrow X \times X$



$$\operatorname{deg}(\Delta \cap T)$$
$$\downarrow$$
$$\sum_i (-1)^i \operatorname{tr}(T \mid H^i(X))$$
$$\sum_\pi \pm \lambda_\pi(T) \cdot \boxed{\dim(H^*(X)[\pi_f])}$$

$\Delta \in \mathrm{Ch}^1(X \times X)_{\mathbb{C}} \circlearrowleft \mathscr{H} \otimes \mathscr{H}$

$\Delta = \sum_\pi \Delta_{\pi \boxtimes \pi^\vee}$

$T = (T \otimes 1)^* \Delta = \sum_\pi \lambda_\pi(T) \Delta_{\pi \boxtimes \pi^\vee}$

$\operatorname{deg}(\Delta \cap T) = (\Delta, T)$

$= \sum_\pi \lambda_\pi(T) \boxed{(\Delta_{\pi \boxtimes \pi^\vee}, \Delta_{\pi \boxtimes \pi^\vee})}$

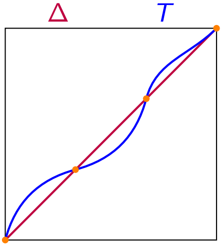
$\rightsquigarrow$ $\# \operatorname{div}(N / \operatorname{cond}(\pi))$

Hecke fixed points: height

Hodge bundle $\mathcal{E}$

$$\begin{array}{ccc} \langle \widehat{\Delta}, \widehat{\mathcal{E}} \boxtimes \widehat{\mathcal{E}}, \widehat{T} \rangle_{\mathcal{X}\times\mathcal{X}} & = & h(\Delta, T) \\ & & \downarrow \\ & & \sum_\pi \lambda_\pi(T) \boxed{h(\Delta_{\pi\boxtimes\pi^\vee}, \Delta_{\pi\boxtimes\pi^\vee})} \end{array}$$

$$\begin{array}{c} \deg(\Delta \cap T) \\ \downarrow \\ \sum_i (-1)^i \operatorname{tr}(T \mid H^i(X)) \\ \sum_\pi \pm\lambda_\pi(T) \cdot \boxed{\dim(H^*(X)[\pi_f])} \end{array}$$



$X \times X$

$\rightsquigarrow$ **Adjoint L-functions**

$\Delta \in \mathrm{Ch}^1(X \times X)_{\mathbb{C}} \circlearrowleft \mathscr{H} \otimes \mathscr{H}$

$\Delta = \sum_\pi \Delta_{\pi\boxtimes\pi^\vee}$

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$\rightsquigarrow \# \operatorname{div}(N / \operatorname{cond}(\pi))$

Adjoint L-functions

$F/\mathbb{Q}$ imag. quad. w/ odd disc. $\quad$ V Hermitian space $\quad$ $X = \mathrm{Sh}_K(G')$ with

$$G' = U(V) \times \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m \quad \text{with} \quad \mathrm{sig}(V) = (n-1, 1).$$

Assume Hecke semisimple for simplicity:

$$\mathrm{Ch}^{n-1}(X \times X)_{\mathbb{C}} = \bigoplus \mathrm{Ch}^{n-1}(X \times X)_{\pi_f \boxtimes \pi'_f}, \quad \Delta = \sum \Delta_{\pi_f \boxtimes \pi_f^\vee} \quad (\Delta_\pi := \Delta_{\pi_f \boxtimes \pi_f^\vee}).$$

Conjecture (C.–Lu–Zhang)

For π a cuspidal automorphic representation of $G = U(V)$ which is cohomological and tempered:

$$h(\Delta_\pi, \Delta_\pi) \sim \frac{L'(0, \pi, \mathrm{Ad})}{L(0, \pi, \mathrm{Ad})} \cdot \dim(H^{n-1}(X)[\pi_f]) \qquad {}^L G \xrightarrow{\mathrm{Ad}} \mathrm{Lie}({}^L G)$$

center at $s = 1/2$

Theorem (C.–Lu–Zhang)

True for compact unitary Shimura curves ($n = 2$) modulo $\sum_{\text{bad red. } p} \bar{\mathbb{Q}} \cdot \log p$.

Cohomology

Cohomology and automorphic L-functions

With $n-1 = \frac{1}{2}\dim(X \times X)$, expect

$$L(s, H^{n-1}(X \times X)) = \prod (\text{automorphic } L\text{-functions}).$$

$$\rightsquigarrow \quad L(s, \pi \boxtimes \pi^\vee, R) = \boxed{L(s, \mathrm{BC}(\pi), \mathrm{As}^+)}\ \boxed{L(s, \mathrm{BC}(\pi), \mathrm{As}^-)}.$$

$$L(s, \pi, \mathrm{Ad}) = L(s, \mathrm{BC}(\pi), \mathrm{As}^{(-1)^n}).$$

Example ($n = 1$)

Consider "$X = \operatorname{Spec} F$"; π automorphic representation for $U(1)$.

$$L(s, H^0(X)) = \zeta_F(s) = \zeta(s) L(s, \mathrm{sgn}_{F/\mathbb{Q}}),$$

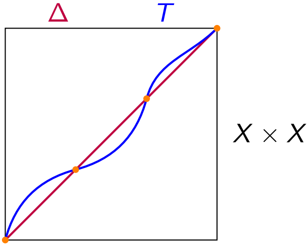
$$\boxed{\zeta(s) = L(s, \mathrm{BC}(\pi), \mathrm{As}^+)}, \quad \boxed{L(s, \mathrm{sgn}_{F/\mathbb{Q}}) = L(s, \mathrm{BC}(\pi), \mathrm{As}^-)}.$$

(Averaged) Colmez: Faltings heights of CM abelian varieties $\sim \frac{L'(0, \mathrm{sgn}_{F/\mathbb{Q}})}{L(0, \mathrm{sgn}_{F/\mathbb{Q}})}$.

Hecke fixed points: counting CM points

$X = X_0(N)$ $\qquad$ $X \xrightarrow{\Delta} X \times X$

Hecke operator $\qquad$ $T \longrightarrow X \times X$



Ex: $p \nmid N$, $\Delta \cap T_p$ "=" $\sum_{\text{char poly } \Phi} Z_\Phi + (\text{cusp.})$

$Z_\Phi = \{(E, \text{level}, \gamma) \mid \gamma \circlearrowright E \quad \text{charpoly}(\gamma) = \Phi\}$

$\Phi \in \mathbb{Z}[t]$ of the form $t^2 + (*)t + p$

$\deg Z_\Phi$ " = " $H(\operatorname{disc} \Phi)$ $\quad$ (Hurwitz class number)

geometric $\qquad$ $\sum_\Phi H(\operatorname{disc}\Phi) + (\text{cusp.})$

Eichler–Selberg trace formula $\qquad$ $\deg(\Delta \cap T)$

spectral $\qquad$ $\sum_i (-1)^i \operatorname{tr}(T \mid H^i(X))$

$\sum_\pi \pm\lambda_\pi(T) \cdot \boxed{\dim(H^*(X)[\pi_f])}$

$\Delta \in \mathrm{Ch}^1(X \times X)_{\mathbb{C}} \circlearrowleft \mathscr{H} \otimes \mathscr{H}$

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$= \sum_\pi \lambda_\pi(T) \boxed{(\Delta_{\pi\boxtimes\pi^\vee}, \Delta_{\pi\boxtimes\pi^\vee})}$

$\rightsquigarrow$ $\# \operatorname{div}(N / \operatorname{cond}(\pi))$

Hecke fixed points: heights of CM points

geometric

Heights of CM cycles Z_Φ $\quad \sum_\Phi \boxed{\langle \widehat{Z}_\Phi, \widehat{\mathcal{E}} \rangle} + (\text{cusp.})$

Hodge bundle $\mathcal{E}$

$\langle \widehat{\Delta}, \widehat{\mathcal{E}} \boxtimes \widehat{\mathcal{E}}, \widehat{T} \rangle_{\mathcal{X}\times\mathcal{X}} \quad = \quad h(\Delta, T)$

spectral

$\sum_\pi \lambda_\pi(T) \boxed{h(\Delta_{\pi\boxtimes\pi^\vee}, \Delta_{\pi\boxtimes\pi^\vee})}$

geometric

$\sum_\Phi H(\operatorname{disc}\Phi) + (\text{cusp.})$

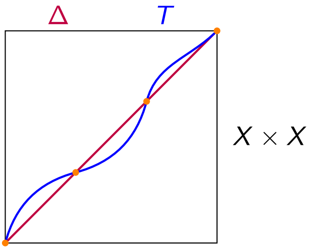
Eichler–Selberg
trace formula

$\deg(\Delta \cap T)$

spectral

$\sum_i (-1)^i \operatorname{tr}(T \mid H^i(X))$

$\sum_\pi \pm\lambda_\pi(T) \cdot \boxed{\dim(H^*(X)[\pi_f])}$



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$\rightsquigarrow$ $\#\operatorname{div}(N/\operatorname{cond}(\pi))$

Strategy: Relative trace formulas

$$
\begin{tikzcd}
\textbf{Heights of CM cycles} \arrow[rr, leftrightarrow, "\text{proof}"] & & \textbf{orbital integrals} \\
h(\Delta, T) = \langle \widehat{\Delta}, \widehat{\mathcal{E}} \boxtimes \widehat{\mathcal{E}}, \widehat{T} \rangle_{\mathcal{X} \times \mathcal{X}} \arrow[u, "\text{geometric}"] \arrow[d, "\text{spectral}"'] & & \begin{array}{c}\text{Relative trace formula} \\ (\mathrm{Res}_{F/\mathbb{Q}}\, GL_n,\ GL_n,\ GL_n)\end{array} \arrow[u, "\text{geometric}"'] \arrow[d, "\text{spectral}"] \\
\sum_{\pi} \lambda_\pi(T)\, h(\Delta_\pi, \Delta_\pi) \arrow[rr, leftrightarrow, "\text{Thm}"] & & \sum_{\pi} \lambda_\pi(T)\, \dfrac{L'(0, \pi, \mathrm{Ad})}{L(0, \pi, \mathrm{Ad})}\, \dim(H^{n-1}(X)[\pi_f])
\end{tikzcd}
$$

Analytic side: Twisted Jacquet–Zagier RTF

$$H = GL_n,\;\; H' = \mathrm{Res}_{F/\mathbb{Q}}\, GL_n,\;\; f \in \mathcal{S}(H'(\mathbb{A})),\;\; \varphi \in \mathcal{S}(W(\mathbb{A})),\;\; W = \left\{ \begin{array}{l} \text{length } n \\ \text{row vectors} \end{array} \right\} \circlearrowleft H$$

Distribution

$$K_f(x,y) = \sum_{\gamma \in H'(\mathbb{Q})} f(x^{-1}\gamma y), \qquad \Theta(x,\varphi) = \sum_{u \in W(\mathbb{Q})} \varphi(ux) \qquad \eta := \mathrm{sgn}_{F/\mathbb{Q}}$$

$$J(f,\varphi,s) := \int_{[H]}^{\mathrm{reg}} \int_{[H]} K_f(x,y)\Theta(x,\varphi)|x|^s \eta(x)^n \eta(y)^{n+1}\, dx\, dy.$$

Local orbital integrals

$S_n = \{x \in H' : x\bar{x} = 1\} \circlearrowleft H$ (by conjugation) $\quad S_n(\mathbb{Q}_p) \cong \mathrm{GL}_n(\mathbb{Q}_p)$ for split p.
$(\gamma, u) \in S_n(\mathbb{Q}_p) \times W(\mathbb{Q}_p)$ regular $\quad f \in \mathcal{S}(S_n(\mathbb{Q}_p)) \quad \varphi \in \mathcal{S}(W(\mathbb{Q}_p))$

$$\mathrm{Orb}(\gamma,u,f,\varphi,s) := \int_{H(\mathbb{Q}_p)} f(h^{-1}\gamma h)\varphi(uh)|h|^s\eta(h)\, dh,$$

$$\mathrm{Orb}(\gamma,u,s) := \mathrm{Orb}(\gamma,u,\mathbf{1}_{S_n(\mathbb{Z}_p)},\mathbf{1}_{W(\mathbb{Z}_p)},s) \text{ for } p \text{ unramified},$$

$$\mathrm{Orb}^\natural(\gamma,u,s) = \frac{\mathrm{Orb}(\gamma,u,s)}{(L\text{-factors})(\text{transfer factors})} \qquad \left(\begin{array}{l} = 1 \text{ if } p \text{ unramified, } \gamma \in S_n(\mathbb{Z}_p), \\ \det\begin{pmatrix} u \\ u\gamma \end{pmatrix} \in \mathbb{Z}_p^\times, \text{ if } n = 2 \end{array} \right)$$

If $n = 2$, $\mathrm{Orb}^\natural(\gamma,u,s) = \mathrm{Orb}^\natural(\gamma,u,1-s)$.

Strategy: Relative trace formulas

local

$$\begin{tikzcd}
\text{“Heights of CM } p\text{-divisible groups”} \arrow[rr, leftrightarrow, "\text{local theorems / conjectures}", "\text{“arithmetic fundamental lemma”}"'] & & \dfrac{\mathrm{Orb}^{\natural\prime}(\gamma, u, 0)}{\mathrm{Orb}^{\natural}(\gamma, u, 0)}
\end{tikzcd}$$

$$\Phi \longleftrightarrow \mathrm{charpoly}(\gamma)$$

$$\begin{tikzcd}
\textbf{Heights of CM cycles} \arrow[rr, leftrightarrow, "\text{proof}"] & & \textbf{orbital integrals} \\
h(\Delta, T) = \langle \widehat{\Delta}, \widehat{\mathcal{E}} \boxtimes \widehat{\mathcal{E}}, \widehat{T} \rangle_{\mathcal{X} \times \mathcal{X}} \arrow[u, "\text{geometric}"] \arrow[d, "\text{spectral}"'] & & \left.\dfrac{d}{ds}\right|_{s=0} J(f, \varphi, s) \arrow[u, "\text{geometric}"'] \arrow[d, "\text{spectral}"] \\
\sum_{\pi} \lambda_\pi(T)\, h(\Delta_\pi, \Delta_\pi) \arrow[rr, leftrightarrow, "\text{Thm}"] & & \sum_{\pi} \lambda_\pi(T)\, \dfrac{L'(0, \pi, \mathrm{Ad})}{L(0, \pi, \mathrm{Ad})}\, \dim(H^{n-1}(X)[\pi_f])
\end{tikzcd}$$

Heights and CM lattices

Given (E, Φ) CM algebra E, CM type Φ:

$$\begin{tikzcd}
\left\{ \begin{array}{l} \mathbb{Z}\text{-lattices } \Lambda \text{ w/ CM} \\ \Lambda_{\mathbb{Q}} \circlearrowleft E \text{ rank one} \end{array} \right\} \arrow[rr, "h_{\mathrm{Fal}}"] & & \mathbb{R} \\
\left\{ \begin{array}{l} \text{Ab. var.}/\mathbb{C} \text{ w/ CM} \\ \text{by } (E, \Phi) \end{array} \right\} \arrow[u, "\wr", "H_1(-, \mathbb{Z})"'] & A \arrow[r, mapsto] & h_{\mathrm{Fal}}(A)
\end{tikzcd}$$

Given (E_p, Φ_p) $E_p/\mathbb{Q}_p$ étale algebra of degree $2n$
$\Phi_p \subseteq \mathrm{Hom}(E_p, \mathbb{C}_p)$ a p-adic CM type, i.e. $|\Phi_p| = n$:

$$\begin{tikzcd}
\left\{ \begin{array}{l} \mathbb{Z}_p\text{-lattices } \Lambda \text{ w/ CM} \\ \Lambda_{\mathbb{Q}_p} \circlearrowleft E_p \text{ rank one} \end{array} \right\} \arrow[rr, "h_p"] & & \mathbb{Q} \cdot \log p \\
\operatorname{colim}_{K/\breve{\mathbb{Q}}_p} \left\{ \begin{array}{l} p\text{-div. gp w/ CM}/\operatorname{Spec} \mathcal{O}_K \\ \text{by } (E_p, \Phi_p) \text{ for } [K : \breve{\mathbb{Q}}_p] < \infty \end{array} \right\} \arrow[u, "\wr", "T_p(-)"'] & Y \arrow[r, mapsto] & h_p(Y)
\end{tikzcd}$$

"Heights" for CM p-divisible groups

Given an isogeny of abelian varieties $\Psi\colon A \to B$ over $\mathcal{O}_K$:

$$h_{\mathrm{Fal}}(B) - h_{\mathrm{Fal}}(A) = \tfrac{1}{2}\log\deg\Psi - \tfrac{1}{[K:\mathbb{Q}]}\log\left|\mathrm{coker}(\Psi\colon \mathrm{Lie}\,A \to \mathrm{Lie}\,B)\right|.$$

Suppose $Y_0/\operatorname{Spec}\mathcal{O}_K$ is a p-divisible group with $Y_0 \circlearrowleft \mathcal{O}_{E_p}$ (maximal order).

Definition: Let $\mathrm{Hom}(Y_0, Y) = \Psi \cdot \mathcal{O}_{E_p}$ (free rank one $/\mathcal{O}_{E_p}$). Then

$$h_p(Y) := \tfrac{1}{2}\log\deg\Psi - \tfrac{\log p}{[K:\breve{\mathbb{Q}}_p]}\,\mathrm{length}\,\mathrm{coker}(\Psi : \mathrm{Lie}\,Y_0 \to \mathrm{Lie}\,Y).$$

Proposition: Given an abelian variety A with CM by (E, Φ), we have canonically

$$\begin{tikzcd} h_{\mathrm{Fal}}(A) \arrow[r, equal] & h(E,\Phi) \arrow[d, "\text{Max. order}"', "\text{Colmez}"] & + & \sum_p h_p(T_p(A)) \arrow[d, "\text{“AFL”}"', "\text{Today (average)}"] \\ & L'/L & & \mathrm{Orb}'\,/\,\mathrm{Orb} \end{tikzcd}$$

"Arithmetic fundamental lemma" for heights

V sig. $(1,1)$ Hermitian space for $F/\mathbb{Q}$, $\quad G = U(V)$, $\quad G' = U(V) \times \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$

$X = \mathrm{Sh}_K(G')$, $\quad p \neq 2$ split in $\mathcal{O}_F$, $\quad K_p \subseteq G(\mathbb{Q}_p)$ maximal, $\quad \mathcal{X}_p \xrightarrow{\text{smooth}} \mathrm{Spf}\, \breve{\mathbb{Z}}_p$.

Supersingular Rapoport–Zink uniformization:

$$\hat{\mathcal{X}}_p^{\mathrm{ss}} \sim I(\mathbb{Q}) \backslash \left(\mathcal{N} \times G(\mathbb{A}_f^p)/K^p \right), \quad \mathcal{N} \cong \coprod_i \mathrm{LT}$$

$$\mathrm{LT}(S) = \left\{ Y/S : \substack{\text{deformation of the height 2,} \\ \text{dimension 1, formal } p\text{-div. group}} \right\}, \quad \mathrm{LT} \cong \mathrm{Spf}\, \breve{\mathbb{Z}}_p[\![t]\!]$$

$$I \text{ inner form of } G, \quad I(\mathbb{Q}_p) = D^\times, \quad I(\mathbb{A}_f^p) \cong G(\mathbb{A}_f^p), \quad \mathrm{LT} \circlearrowleft \mathcal{O}_D^\times$$

$$\text{For } g \in \mathcal{O}_D^\times \setminus \mathbb{Z}_p^\times : \quad \mathrm{CM}(g) := \{ Y \in \mathrm{LT} : gY = Y \} \xrightarrow{\text{finite flat}} \mathrm{Spf}\, \breve{\mathbb{Z}}_p$$

Local Theorem (C.–Lu–Zhang)

For $g \leftrightarrow (\gamma, u)$, i.e. $\mathrm{charpoly}(g) = \mathrm{charpoly}(\gamma)$ and (γ, u) regular (here $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$):

$$\deg \mathrm{CM}(g) = \sum_{Y \in \mathrm{CM}(g)(\mathbb{C}_p)} 1 = [\breve{\mathbb{Q}}_p(g) : \breve{\mathbb{Q}}_p] \cdot \mathrm{Orb}^\natural(\gamma, u, 0),$$

$$h_p(\mathrm{CM}(g)) = \sum_{Y \in \mathrm{CM}(g)(\mathbb{C}_p)} h_p(Y) = -\tfrac{1}{2} [\breve{\mathbb{Q}}_p(g) : \breve{\mathbb{Q}}_p] \cdot \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}^\natural(\gamma, u, s).$$

Archimedean: unfolding

Harmonic Green function $\mathcal{G}_T$ for $T \to X \times X$, i.e. "log potential"

$$\mathcal{G}_T = -\log|f|^2 + \text{smooth} \quad \text{where} \quad T \text{ is locally } \{f = 0\}.$$

For *Eisenstein* $T \in \mathscr{H}$ (annihilating $H^0(X_{\mathbb{C}}, \mathbb{C})$):

$$h(\Delta, T) \overset{\text{Arch.}}{\rightsquigarrow} \int_{\Delta(\mathbb{C})} \mathcal{G}_T(z, z) \frac{dx\,dy}{y^2}$$

$$\overset{\text{unfolding}}{\rightsquigarrow} \lim_{\nu \to 1^+} \sum_g \boxed{\mathrm{Orb}_{U(1,1)(\mathbb{R})}(g, Q_{\nu-1})} \cdot \begin{pmatrix} \text{non-Archimedean} \\ \text{orbital integrals for } T \end{pmatrix}$$

For $t > 1$, we have

$$Q_{\nu-1}(t) := \int_0^\infty (t + \sqrt{t^2 - 1}\cosh u)^{-\nu}\, du \quad (\textbf{singularity} \text{ as } t \to 1)$$

$$\underset{\text{decomp.}}{\overset{\text{Cartan}}{\rightsquigarrow}} K_\infty \backslash U(1,1)(\mathbb{R}) / K_\infty \dashrightarrow \mathbb{R} \quad (\textbf{singularity} \text{ along } K_\infty).$$

Archimedean: orbital integrals

$$\mathrm{Orb}_{U(1,1)(\mathbb{R})}(g, Q_{\nu-1}) \overset{?}{\longleftrightarrow} \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}^{\natural}_{\mathrm{GL}_2(\mathbb{R})}(s) \text{ on } \mathrm{GL}_2(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{C})/\mathrm{GL}_2(\mathbb{R})$$

$$\mathbf{1}_{U(2)(\mathbb{R})} \xrightarrow{\text{MSY transfer}} \varphi \implies \mathrm{Orb}^{\natural}_{\mathrm{GL}_2(\mathbb{R})}(\varphi, \gamma, u, s)\Big|_{s=0} = \begin{cases} 1 & \text{if } \gamma \text{ is elliptic} \\ 0 & \text{if } \gamma \text{ is hyperbolic} \end{cases}$$

"Derivative-to-special-value transfer" (C.–Lu–Zhang)

There exists $\phi_{\text{Deriv}} \colon K_\infty \backslash U(1,1)(\mathbb{R})/K_\infty \dashrightarrow \mathbb{R}$ (**singularity** along K_∞) such that for $g \leftrightarrow (\gamma, u)$,

$$\mathrm{Orb}_{U(1,1)(\mathbb{R})}(g, \phi_{\text{Deriv}}) = \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}^{\natural}_{\mathrm{GL}_2(\mathbb{R})}(\varphi, \gamma, u, s).$$

$$\mathrm{Ei}(-t) := -\int_1^\infty e^{-tx}x^{-1}\,dx \overset{\text{Cartan decomp.}}{\rightsquigarrow} \phi_{\mathrm{Deriv}}$$

$$\mathrm{Orb}_{U(1,1)(\mathbb{R})}(g, Q_{\nu-1}) \overset{?}{\longleftrightarrow} \mathrm{Orb}_{U(1,1)(\mathbb{R})}(g, \mathrm{Ei})$$

Archimedean: spectral decomposition

$$\mathrm{Orb}_{U(1,1)(\mathbb{R})}(g, Q_{\nu-1}) \overset{?}{\longleftrightarrow} \mathrm{Orb}_{U(1,1)(\mathbb{R})}(g, \mathrm{Ei})$$

Singularity cancellation:

$$\phi_{\mathrm{diff},\nu} := 2Q_{\nu-1}(1+t) + \mathrm{Ei}(-2\pi t) \quad \text{(extends continuously to } t = 0\text{)}$$

so $\phi_{\mathrm{diff},\nu} \in C^0(K_\infty \backslash U(1,1)(\mathbb{R})/K_\infty)$.

By the Selberg trace formula:

$$\lim_{\nu \to 1^+} \sum_g \mathrm{Orb}_{U(1,1)(\mathbb{R})}(g, \phi_{\mathrm{diff},\nu}) \cdot \begin{pmatrix} \text{non-Archimedean} \\ \text{orbital integrals for } T \end{pmatrix}$$
$$= \sum_{\substack{\pi \subseteq L^2([U(V)]) \\ \dim \pi = \infty, \quad \pi_\infty^{K_\infty} \neq 0}} \mathrm{tr}(\pi(\phi_{\mathrm{diff},\nu=1} \otimes T)).$$

(not cohomological of middle degree)